Appendix C Special functions

C.1 Modified Bessel functions

The modified Bessel function of the second kind with order parameter $\alpha \in \mathbb{R}$ admits the Fourier-based representation [AS72]

$$K_\alpha(\omega) = \int_{\mathbb{R}} \frac{\mathrm{e}^{-\mathrm{j}\omega x}}{(1+x^2)^{|\alpha|}}\,\mathrm{d}x.$$

It has the property that $K_\alpha(x) = K_{-\alpha}(x)$. A special case of interest is $K_{\frac{1}{2}}(x) = \left(\frac{\pi}{2x}\right)^{\frac{1}{2}} \mathrm{e}^{-x}$. The small scale behavior of $K_\alpha(x)$ is $K_\alpha(x) \sim \frac{\Gamma(\alpha)}{2}\left(\frac{2}{x}\right)^\alpha$ as $x \to 0$. In order to determine the form of the variance-gamma distribution around the origin, we can rely on the following expansion which includes a few more terms:

$$\begin{aligned} K_\alpha(x) =& x^{-\alpha}\left(2^{\alpha-1}\Gamma(\alpha) - \frac{2^{\alpha-3}\Gamma(\alpha)x^2}{\alpha-1} + O\left(x^4\right)\right) \\ &+ x^{\alpha}\left(2^{-\alpha-1}\Gamma(-\alpha) + \frac{2^{-\alpha-3}\Gamma(-\alpha)x^2}{\alpha+1} + O\left(x^4\right)\right). \end{aligned}$$

At the other end of the scale, its asymptotic behavior is

$$K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x} \text{ as } x \to +\infty.$$

C.2 Gamma function

Euler's gamma function constitutes an analytic extension of the factorial function $n! = \Gamma(n+1)$ to the complex plane. It is defined by the integral

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t,$$

which is convergent for $\mathrm{Re}(z) > 0$. Specific values are $\Gamma(1) = 1$ and $\Gamma(1/2) = \sqrt{\pi}$. The gamma function satisfies the functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{C.1}$$

which is compatible with the recursive definition of the factorial $n! = n(n-1)!$. Another useful result is Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

By combining the above with (C.1), we obtain

$$\mathrm{sinc}(z) = \frac{\sin(\pi z)}{\pi z} = \frac{1}{\Gamma(1-z)\,\Gamma(1+z)}, \tag{C.2}$$

which makes an intriguing connection with the *sinus cardinalis* function. There is a similar link with Euler's beta function

$$\begin{aligned} B(z_1, z_2) &= \int_0^1 t^{z_1-1}(1-t)^{z_2-1}\,\mathrm{d}t \\ &= \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} \end{aligned} \tag{C.3}$$

with $\mathrm{Re}(z_1), \mathrm{Re}(z_2) > 0$.

$\Gamma(z)$ also admits the well-known product decomposition

$$\Gamma(z) = \frac{\mathrm{e}^{-\gamma_0 z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} \mathrm{e}^{z/n} \tag{C.4}$$

where γ_0 is the Euler-Mascheroni constant. The above allows us to derive the expansion

$$-\log|\Gamma(z)|^2 = 2\gamma_0 \mathrm{Re}(z) + \log|z|^2 + \sum_{n=1}^{\infty} \left(\log\left|1 + \frac{z}{n}\right|^2 - 2\frac{\mathrm{Re}(z)}{n}\right),$$

which is directly applicable to the likelihood function associated with the Meixner distribution. Also relevant to that context is the integral relation

$$\int_{\mathbb{R}} \left|\Gamma(\frac{r}{2} + \mathrm{j}x)\right|^2 \mathrm{e}^{\mathrm{j}zx}\,\mathrm{d}x = 2\pi\Gamma(r)\left(\frac{1}{2\cosh\frac{z}{2}}\right)^r$$

for $r > 0$ and $z \in \mathbb{C}$, which can be interpreted as a Fourier transform by setting $z = -\mathrm{j}\omega$.

Euler's digamma function is defined as

$$\psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{C.5}$$

while its mth order derivative

$$\psi^{(m)}(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}}\log\Gamma(z) \tag{C.6}$$

is called the polygamma function of order m.

C.3 Symmetric-alpha-stable distributions

The SαS pdf of degree $\alpha \in (0,2]$ and scale parameter s_0 is best defined via its characteristic function

$$p(x;\alpha,s_0) = \int_{\mathbb{R}} \mathrm{e}^{-|s_0\omega|^\alpha}\mathrm{e}^{\mathrm{j}\omega x}\frac{\mathrm{d}\omega}{2\pi}.$$

Alpha-stable distributions do not admit closed-form expressions, except for the special cases $\alpha = 1$ (Cauchy) and 2 (Gauss distribution). Moreover, their absolute moments of order p, $\mathbb{E}\{|X|^p\}$, are unbounded for $p > \alpha$, which is characteristic of heavy-tailed distributions. We can relate the (symmetric) γth-order moments of their characteristic function to the gamma function by performing the change of variable $t = (s_0\omega)^\alpha$, which leads to

$$\int_{\mathbb{R}} |\omega|^\gamma \mathrm{e}^{-|s_0\omega|^\alpha}\,\mathrm{d}\omega = 2\int_0^\infty \frac{s_0^{-\gamma-1}}{\alpha} t^{\frac{\gamma-\alpha+1}{\alpha}} \mathrm{e}^{-t}\,\mathrm{d}t = 2\frac{s_0^{-\gamma-1}\Gamma\left(\frac{\gamma+1}{\alpha}\right)}{\alpha}. \tag{C.7}$$

By using the correspondence between Fourier-domain moments and time-domain derivatives, we use this result to write the Taylor series of $p(x;\alpha,s_0)$ around $x = 0$ as

$$p(x;\alpha,s_0) = \sum_{k=0}^{\infty} \frac{s_0^{-2k-1}}{\pi\alpha}\Gamma\left(\frac{2k+1}{\alpha}\right)(-1)^k \frac{|x|^{2k}}{(2k)!}, \tag{C.8}$$

which involves even terms only (because of symmetry). The moment formula (C.7) also yields a simple expression for the slope of the score at the origin, which is given by

$$\Phi_X''(0) = -\frac{p_X''(0)}{p_X(0)} = \frac{\Gamma\left(\frac{3}{\alpha}\right)}{s_0^2\Gamma\left(\frac{1}{\alpha}\right)}.$$

Similar techniques are applicable to obtain the asymptotic form of $p(x;\alpha,s_0)$ as x tends to infinity [Ber52, TN95]. To characterize the tail behavior, it is sufficient to consider the first term of the asymptotic expansion

$$p(x;\alpha,s_0) \sim \frac{1}{\pi}\Gamma(\alpha+1)\sin\left(\frac{\pi\alpha}{2}\right)s_0^\alpha \frac{1}{|x|^{\alpha+1}} \text{ as } x \to \pm\infty, \tag{C.9}$$

which emphasizes the algebraic decay of order $(\alpha+1)$ at infinity.