Appendix C Special functions

C.1 Modified Bessel functions

The modified Bessel function of the second kind with order parameter \( \alpha \in \mathbb{R} \) admits the Fourier-based representation [AS72]

\[
K_\alpha(x) = \int_{\mathbb{R}} \frac{e^{-|\nu|}}{(1 + |\nu|^2)^{\alpha/2}} d\nu.
\]

It has the property that \( K_\alpha(x) = K_{-\alpha}(x) \). A special case of interest is \( K_{1/2}(x) = \sqrt{\pi} x^{-1} e^{-x} \).

The small scale behavior of \( K_\alpha(x) \) is \( K_\alpha(x) \sim \sqrt{2\pi} x^{\alpha - 1/2} e^{-x} \) as \( x \to 0 \). In order to determine the form of the variance-gamma distribution around the origin, we can rely on the following expansion which includes a few more terms:

\[
K_\alpha(x) = x^{-\alpha} \left( 2^{\alpha-1} \Gamma(\alpha) - \frac{2^{\alpha-3} \Gamma(\alpha) x^2}{\alpha - 1} + O(x^4) \right) + x^\alpha \left( 2^{\alpha-1} \Gamma(-\alpha) + \frac{2^{\alpha-3} \Gamma(-\alpha) x^2}{\alpha + 1} + O(x^4) \right).
\]

At the other end of the scale, its asymptotic behavior is

\[
K_\alpha(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x} \text{ as } x \to +\infty.
\]

C.2 Gamma function

Euler’s gamma function constitutes an analytic extension of the factorial function \( n! = \Gamma(n + 1) \) to the complex plane. It is defined by the integral

\[
\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt,
\]

which is convergent for \( \text{Re}(z) > 0 \). Specific values are \( \Gamma(1) = 1 \) and \( \Gamma(1/2) = \sqrt{\pi} \). The gamma function satisfies the functional equation

\[\Gamma(z + 1) = z \Gamma(z) \quad (C.1)\]
which is compatible with the recursive definition of the factorial \( n! = n(n-1)! \). Another useful result is Euler’s reflection formula

\[
\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},
\]

By combining the above with (C.1), we obtain

\[
\text{sinc}(z) = \frac{\sin(\pi z)}{\pi z} = \frac{1}{\Gamma(1-z)\Gamma(1+z)}, \tag{C.2}
\]

which makes an intriguing connection with the \textit{sinus cardinalis} function. There is a similar link with Euler’s beta function

\[
B(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1} \, dt \tag{C.3}
\]

with \( \Re(z_1), \Re(z_2) > 0 \).

\( \Gamma(z) \) also admits the well-known product decomposition

\[
\Gamma(z) = e^{-\gamma_0 z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \tag{C.4}
\]

where \( \gamma_0 \) is the Euler-Mascheroni constant. The above allows us to derive the expansion

\[
-\log|\Gamma(z)|^2 = 2\gamma_0 \Re(z) + \log|z|^2 + \sum_{n=1}^{\infty} \left( \log \left|1 + \frac{z}{n}\right|^2 - 2 \frac{\Re(z)}{n} \right),
\]

which is directly applicable to the likelihood function associated with the Meixner distribution. Also relevant to that context is the integral relation

\[
\int_{\mathbb{R}} \left| \Gamma(\frac{r}{2} + jx) \right|^2 e^{jux} \, dx = 2\pi \Gamma(r) \left( \frac{1}{2\cosh \frac{x}{2}} \right)^r
\]

for \( r > 0 \) and \( z \in \mathbb{C} \), which can be interpreted as a Fourier transform by setting \( z = -j\omega \).

Euler’s digamma function is defined as

\[
\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \tag{C.5}
\]

while its \( m \)th order derivative

\[
\psi^{(m)}(z) = \frac{d^{m+1}}{dz^{m+1}} \log \Gamma(z) \tag{C.6}
\]

is called the polygamma function of order \( m \).

### C.3 Symmetric-alpha-stable distributions

The \textit{S\&S} pdf of degree \( \alpha \in (0,2] \) and scale parameter \( s_0 \) is best defined via its characteristic function

\[
p(x; \alpha, s_0) = \int_{\mathbb{R}} e^{-|s_0\omega|^\alpha} e^{j\omega x} \frac{d\omega}{2\pi}.
\]
Alpha-stable distributions do not admit closed-form expressions, except for the special cases \( \alpha = 1 \) (Cauchy) and 2 (Gauss distribution). Moreover, their absolute moments of order \( p, \mathbb{E}|X|^p \), are unbounded for \( p > \alpha \), which is characteristic of heavy-tailed distributions. We can relate the (symmetric) \( \gamma \)-th-order moments of their characteristic function to the gamma function by performing the change of variable \( t = (s_0 \omega)^{\gamma} \), which leads to

\[
\int_{\mathbb{R}} |\omega|^\gamma e^{-|s_0 \omega|^\alpha} d\omega = 2 \int_{0}^{\infty} \frac{s_0^{-\gamma-1}}{\alpha} t^{\gamma-\alpha+1} e^{-t} dt = 2 \frac{s_0^{-\gamma-1} \Gamma(\frac{\gamma+1}{\alpha})}{\alpha}. \quad (C.7)
\]

By using the correspondence between Fourier-domain moments and time-domain derivatives, we use this result to write the Taylor series of \( p(x; \alpha, s_0) \) around \( x = 0 \) as

\[
p(x; \alpha, s_0) = \sum_{k=0}^{\infty} \frac{s_0^{-2k-1}}{\pi \alpha} \Gamma\left(\frac{2k+1}{\alpha}\right) (-1)^k |x|^{2k} \left(\frac{2}{k!}\right), \quad (C.8)
\]

which involves even terms only (because of symmetry). The moment formula (C.7) also yields a simple expression for the slope of the score at the origin, which is given by

\[
\Phi'_X(0) = -\frac{p'_X(0)}{p_X(0)} = \frac{\Gamma\left(\frac{\alpha}{2}\right)}{s_0^\alpha \Gamma\left(\frac{1}{\alpha}\right)}.
\]

Similar techniques are applicable to obtain the asymptotic form of \( p(x; \alpha, s_0) \) as \( x \) tends to infinity [Ber52, TN95]. To characterize the tail behavior, it is sufficient to consider the first term of the asymptotic expansion

\[
p(x; \alpha, s_0) \sim \frac{\Gamma(\alpha+1)}{\pi} \sin\left(\frac{\pi \alpha}{2}\right) s_0^\alpha |x|^{\alpha+1} \frac{1}{|x|^{\alpha+1}} \text{ as } x \to \pm \infty, \quad (C.9)
\]

which emphasizes the algebraic decay of order \((\alpha + 1)\) at infinity.